

# Gluing of Surfaces with Polygonal Boundaries

*E.T.Akhmedov and Sh.Shakirov*

ITEP, B.Chermushkinskaya, 25, Moscow, Russia 117218  
Moscow Institute of Physics and Technology, Dolgoprudny, Russia

## ABSTRACT

By pairwise gluing of edges of a polygon, one produces two-dimensional surfaces with handles and boundaries. In this paper, we count the number  $\mathcal{N}_{g,L}(n_1, n_2, \dots, n_L)$  of different ways to produce a surface of given genus  $g$  with  $L$  polygonal boundaries with given numbers of edges  $n_1, n_2, \dots, n_L$ . Using combinatorial relations between graphs on real two-dimensional surfaces, we derive recursive relations between  $\mathcal{N}_{g,L}$ . We show that Harer-Zagier numbers appear as a particular case of  $\mathcal{N}_{g,L}$  and derive a new explicit expression for them.

## 1 Introduction

A classical question in enumerative combinatorics is: how many ways are there to glue pairwise all edges of a  $2N$ -gon so as to produce a surface of given genus  $g$ ? Such a (*complete*) gluing means that one performs exactly  $N$  contractions, so that no edges of a polygon remain unglued. This problem has been solved in [1] and the answer is given by the so called Harer–Zagier numbers  $\epsilon_g(N)$

$\epsilon_g(N)$	$g = 0$	$g = 1$	$g = 2$	$\dots$
$N = 1$	1	–	–	–
$N = 2$	2	1	–	–
$N = 3$	5	10	–	–
$N = 4$	14	70	21	–
$N = 5$	42	420	483	–
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

with the general formula:

$$\epsilon_g(N) = \frac{(2N)!}{(N+1)!(N-2g)!} \times \text{Coefficient of } x^{2g} \text{ in } \left( \frac{x/2}{\tanh x/2} \right)^{N+1}. \quad (1)$$

In this paper, we study more general *incomplete* gluings, i.e, we allow some edges of a polygon to remain unglued. In this way we generalize the problem and, as a result, we obtain a simple recursion relation for the corresponding numbers of the ways to glue.

First of all, let us give a definition of gluing. A particular gluing of a polygon's boundary is defined via a semi-Gaussian word along its boundary. A semi-Gaussian word is such a sequence of letters in which every entry appears *no more* than two times. This sequence of letters is assigned to the boundary of the polygon in such a way that one letter corresponds to one edge at the boundary. If some letter appears twice in the semi-Gaussian word, then we identify corresponding two edges at the boundary of the polygon in a way which respects the orientability of the two-dimensional surface<sup>1</sup>. Thus, a complete gluing is such a word, in which every entry appears exactly two times.

After the gluing — i.e, after identifying all the edges of a polygon that correspond to the repeated letters — we arrive at the orientable two-dimensional surface, which in general has polygonal boundaries. At the boundaries we have un-glued edges corresponding to the letters which appear only one time in the corresponding semi-Gaussian word. We can have 0-gons (punctured points), 1-gons, 2-gons, 3-gons (triangles) and etc. among the boundaries. Note, that if one deals with complete gluings, there can be only punctured points.

It is worth mentioning now that in this paper we distinguish all the non-repeated letters in the semi-Gaussian word and, hence, we distinguish all the edges of the polygonal boundaries of the resulting two-dimensional surfaces. As well we always consider such semi-Gaussian words which correspond to the gluings leading to the surfaces with some fixed in advance sequence of letters on the edges of the polygonal boundaries.

<sup>1</sup>In this paper we always consider orientable two-dimensional surfaces.

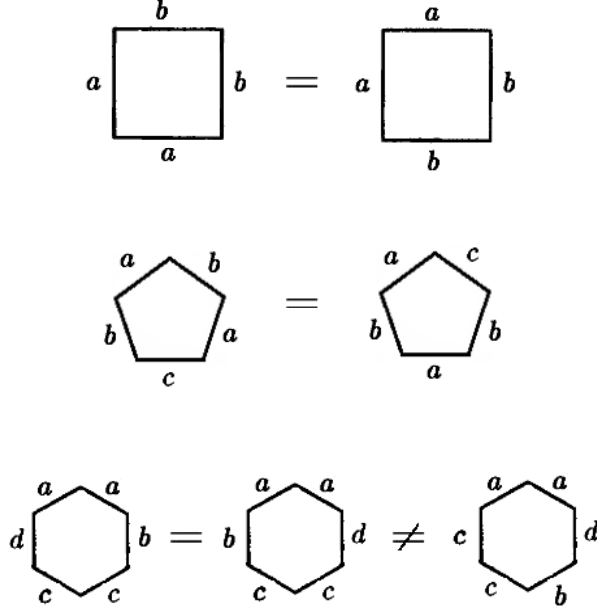


Figure 1: Several examples of equivalent and non-equivalent gluings.

Now, we consider two gluings of a polygon as equivalent if the corresponding semi-Gaussian words can be mapped to each other via a rotation of the polygon and/or via a re-naming of the repeated letters in the words (see fig.1). Note that such an identification of the gluings is different from the one adopted in [1]. We discuss this difference in a greater detail in the third section.

We use such an equivalence relation for the gluings, because it corresponds to the natural identification of graphs on two-dimensional surfaces. Obviously, the image of the polygon's boundary after the gluing is a graph, embedded into the surface. The edges of the graph either belong to the polygonal boundaries or join the vertices belonging to different components of the boundary. Edges of the graph do not intersect each other, unless they end at the same vertex of the boundary.

Consider now graphs on two-dimensional orientable surfaces. We consider two such graphs as equivalent if they can be mapped onto each other via a diffeomorphism of the surface which respects its orientation and acts trivially at the boundary (but does not have to be isotopic to a trivial diffeomorphism). Cutting a surface along an embedded graph, one will obtain a collection of two-dimensional surfaces. In this paper we are interested only in *one-face graphs*: it is such a graph, cutting the surface along which, one obtains a single polygon without any graphs drawn on it (except the one going along its boundary). Thus, we arrive at the statement that there is a one-to-one map between the set of equivalence classes of all possible gluings of all possible polygons and the set of equivalence classes of all possible one-face graphs on all possible two-dimensional surfaces. Such a one-to-one correspondence is, in fact, a property of the identification of gluings, adopted in our paper.

The question that we address is: how many ways are there to glue edges of an  $N$ -gon so as to produce a surface of given genus  $g$  with  $L$  polygonal boundaries with given numbers of edges  $n_1, n_2, \dots, n_L$ ? We denote this number as  $\mathcal{N}_{g,L}(n_1, n_2, \dots, n_L)$ .

As one can see,  $\mathcal{N}$  is expressed in terms of the resulting surface, that is,  $g$ ,  $L$  and  $n_1, \dots, n_L$ . One naturally expects  $\mathcal{N}_{g,L}$  to be a symmetric function of  $n_1, n_2, \dots, n_L$  because the enumeration of holes is arbitrary. Unlike [1], we do not treat the total number  $N$  of edges of the glued  $N$ -gon as an independent parameter: in fact, it can be easily seen that  $N = n_1 + n_2 + \dots + n_L + 4g + 2L - 2$ . Obviously, in our case  $N$  is not necessarily even, namely, we glue  $N$ -gon rather than  $2N$ -gon.

In this paper we find an explicit formula for  $\mathcal{N}_{g,L}(n_1, n_2, \dots, n_L)$ . We find  $\mathcal{N}_{g,L}$  inductively, via a recursive relation which expresses the number of one-face graphs on the surface with  $L$  holes and  $g$  handles through the numbers of one-face graphs on the surface with fewer holes or fewer handles and more holes. This relation has a clear combinatorial origin following from consecutive cutting of the surface along the edges of the surface graph.

As one could expect,  $\mathcal{N}_{g,L}(n_1, n_2, \dots, n_L)$  reduces to  $\epsilon_g(N)$  in certain limit. It produces several explicit formulas for  $\epsilon_g(N)$ , which (to the best of our knowledge) were not known before.

## 2 The calculation of $\mathcal{N}_{g,L}(n_1, n_2, \dots, n_L)$

We count the number of different gluings by means of counting the number of different graphs. We consider a surface with  $g$  handles, which is equipped with  $L$  polygonal boundaries, with numbers of edges  $n_1, n_2, \dots, n_L$ . To begin, we assume that all  $n_i > 0$ . The case when some  $n_i = 0$  will be considered later.

**Lemma I.** If all  $n_i > 0$ , then  $\mathcal{N}$  satisfies the following recursive equation:

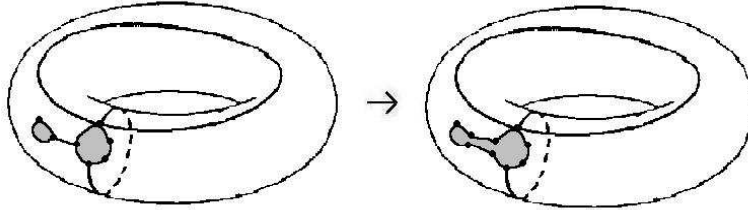
$$(L + 2g - 1) \cdot \mathcal{N}_{g,L}(n_1, n_2, \dots, n_L) = \sum_{i < j} n_i n_j \cdot \mathcal{N}_{g,L-1}(n_i + n_j + 2, n_1, \dots, \tilde{n}_i, \dots, \tilde{n}_j, \dots, n_L) \\ + 1/2 \sum_i n_i \cdot \sum_{x=1}^{n_i+1} \mathcal{N}_{g-1,L+1}(n_i + 2 - x, x, n_1, \dots, \tilde{n}_i, \dots, n_L), 1 \leq i, j \leq L. \quad (2)$$

The initial condition for the recursion is that  $\mathcal{N}_{0,1}(n) = \text{const} = 1$ , which is obvious, because a sphere with one hole is already a polygon. If  $L \leq 0$  or  $g < 0$ , it is assumed that  $\mathcal{N} = 0$ . The check-sign  $\tilde{n}$  stands for omitting of the arguments.

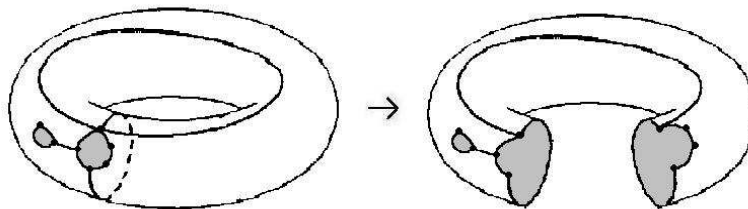
**Proof.** The recursion relation we consider and its derivation are similar to the ones given in [1] (see page 475).  $\mathcal{N}_{g,L}$  counts the number of one-face graphs. All one-face graphs on the same surface consist of the same number of edges. It can be easily seen that there are  $L + 2g - 1$  edges that connect boundary vertices and do not belong to the boundary. Let us take a graph from the family  $\mathcal{N}_{g,L}$  and cut the surface along one of such edges. By the cutting we mean that we unglue two of the edges of the polygon and assign different letters to them. I.e., we distinguish two new edges of the resulting surface.

Number  $\mathcal{N}_{g,L}$  of one-face graphs on original surface is proportional to the sum over the different edges of the numbers of one-face graphs on the surfaces, obtained by cutting along every edge. The coefficient of the proportionality is  $L + 2g - 1$ , because by doing this, we count one graph on the original surface exactly  $L + 2g - 1$  times. This explains the left hand side of (2).

Via the cutting along an edge of a graph from the class  $\mathcal{N}_{g,L}$  we obtain either graph from the class  $\mathcal{N}_{g,L-1}$  (i.e. the cutting decreases the number of holes) or from the class  $\mathcal{N}_{g-1,L+1}$  (i.e. the cutting decreases the number of handles and increase the number of holes). The latter two classes correspond to the two terms on the right hand side of (2). What is the same, edges of the one-face graphs are divided into two classes: connecting two different boundaries, e.g.



and connecting one boundary with itself, e.g.



These two classes correspond to the first and the second terms on the right hand side of (2).

- To connect two different (e.g.  $i$ -th and  $j$ -th) boundaries, one can draw an edge from one of  $n_i$  vertices of the first boundary to one of the  $n_j$  vertices of the second boundary. Hence, there are  $n_i n_j$  non-diffeomorphic to each other ways to draw one edge connecting the two boundaries in question. By cutting over such an edge, we obtain a surface with fewer holes and the same genus. Two holes with  $n_i$  and  $n_j$  edges transform into a single hole with  $n_i + n_j + 2$  edges (see the first figure). This observation leads to the first term on the right hand side of (2).

- To connect an  $i$ -th boundary with itself, we have to draw one of the edge encircling one of  $g$  handles (otherwise, this will fail to be a one-face graph). The cut over such an edge eliminates the corresponding handle and transforms the hole into two holes (see the second figure). The sum of the numbers of edges on the new holes is equal to  $n_i + 2$ , because two new edges appear after performing the cut.

The boundary's vertices are enumerated from 1 to  $n_i$ , in a natural order. Suppose that the edge terminates in the  $p$ -th and  $q$ -th vertices on the boundary. Such an edge divides the boundary into two parts. Numbers of edges on these two parts are equal to  $|p - q|$  and  $n_i - |p - q|$ . After performing the cut, these two parts transform into two independent holes on the cutted surface. These holes will have  $|p - q| + 1$  and  $n_i - |p - q| + 1$  edges. One finally takes a sum over all edges, that is, over  $1 \leq p \leq q \leq n_i$ :

$$\begin{aligned} & \sum_{p \leq q} \mathcal{N}_{g-1, L+1}(|p - q| + 1, n_i - |p - q| + 1, n_1, \dots, \tilde{n}_i, \dots, n_L) = \\ & 1/2 \sum_{p=1}^{n_i} \sum_{q=1}^{n_i} \mathcal{N}_{g-1, L+1}(|p - q| + 1, n_i - |p - q| + 1, n_1, \dots, \tilde{n}_i, \dots, n_L) + \\ & + 1/2 \sum_{p=q=1}^{n_i} \mathcal{N}_{g-1, L+1}(|p - q| + 1, n_i - |p - q| + 1, n_1, \dots, \tilde{n}_i, \dots, n_L). \end{aligned}$$

This expression is simplified by introducing a new summation variable  $x = |p - q| + 1$ . Then, it turns into

$$\begin{aligned} & n_i/2 \sum_{x=1}^{n_i} \mathcal{N}_{g-1, L+1}(x, n_i + 2 - x, n_1, \dots, \tilde{n}_i, \dots, n_L) + n_i/2 \mathcal{N}_{g-1, L+1}(1, n_i + 1, n_1, \dots, \tilde{n}_i, \dots, n_L) = \\ & n_i/2 \cdot \sum_{x=1}^{n_i+1} \mathcal{N}_{g-1, L+1}(x, n_i + 2 - x, n_1, \dots, \tilde{n}_i, \dots, n_L). \end{aligned}$$

This calculation justifies the second term of the right hand side of (2).

The lemma is proved.

**Theorem I.** The unique solution of (2) is given by the following explicit formula:

$$\mathcal{N}_{g, L}(n_1, \dots, n_L) = \frac{1}{4^g} \cdot n_1 \dots n_L \cdot \frac{(\Sigma n + 4g + 2L - 3)!}{(\Sigma n + 2g + L - 1)!} \cdot \sum_{\lambda_1 + \dots + \lambda_L = g} \prod_{k=1}^L \frac{(2\lambda_k + n_k)!}{(n_k)!(2\lambda_k + 1)!} \quad (3)$$

where the sum is taken over all ordered decompositions  $g = \lambda_1 + \dots + \lambda_L$  of a non-negative integer number  $g$  into  $L$  non-negative integers  $\lambda_1, \dots, \lambda_L$ . We use the notation  $\Sigma n = n_1 + \dots + n_L$ .

**Proof.** The theorem is proved straightforwardly by induction.

Let us give a proper generalization of (2) and (3) to the case when some of  $n_i$ 's can be zero. Actually, in this paper, we always assume that there is at least one unglued edge, i.e. we always consider  $\sum_{i=1}^L n_i \geq 1$ . Our formulas do not work for the case when all  $n_i$  are equal to zero, i.e.  $\mathcal{N}_{g, L}(0, 0, \dots, 0)$ .

**Lemma II.** If some (but not all)  $n_i$  are zero, then  $\mathcal{N}$  satisfies the following recursive equation:

$$\begin{aligned} (L + 2g - 1) \cdot \tilde{\mathcal{N}}_{g, L}(n_1, n_2, \dots, n_L) &= \sum_{i < j} \tilde{n}_i \tilde{n}_j \cdot \tilde{\mathcal{N}}_{g, L-1}(n_i + n_j + 2, n_1, \dots, \tilde{n}_i, \dots, \tilde{n}_j, \dots, n_L) \\ &+ 1/2 \sum_i \tilde{n}_i \cdot \sum_{x=1}^{n_i+1} \tilde{\mathcal{N}}_{g-1, L+1}(n_i + 2 - x, x, n_1, \dots, \tilde{n}_i, \dots, n_L), \end{aligned} \quad (4)$$

where  $\mathcal{N}_{g, L}(n_1, n_2, \dots, n_L) = \tilde{\mathcal{N}}_{g, L}(n_1, n_2, \dots, n_L) / \#(n_i = 0)!$  and  $\tilde{n} = n$  if  $n > 0$  and  $\tilde{n} = 1$  if  $n = 0$ . Here  $\#(n_i = 0)$  is the number of  $n_i$ 's which are equal to zero.

**Proof.** The proof repeats that of the Lemma I, with minor corrections. A boundary with 0 edges is a punctured point. It follows immediately, that one can draw an edge in  $\tilde{n}_i \tilde{n}_j$  ways to connect two different ( $i$ -th and  $j$ -th) boundaries. Finally, it is customary to assume punctured points to be indistinguishable. Therefore, to avoid counting equivalent graphs several times, one should divide over the factorial of the number of punctured points, that is  $\#(n_i = 0)!$ .

**Theorem II.** The unique solution of (4) is given by the following explicit formula:

$$\mathcal{N}_{g,L}(n_1, \dots, n_L) = \frac{1}{4^g} \frac{1}{\#(n_i = 0)!} \cdot \tilde{n}_1 \dots \tilde{n}_L \cdot \frac{(\Sigma n + 4g + 2L - 3)!}{(\Sigma n + 2g + L - 1)!} \cdot \sum_{\lambda_1 + \dots + \lambda_L = g} \prod_{k=1}^L \frac{(2\lambda_k + n_k)!}{(n_k)!(2\lambda_k + 1)!}, \quad (5)$$

where the sum is taken over all ordered decompositions  $g = \lambda_1 + \dots + \lambda_L$  of the integer number  $g$  into  $L$  non-negative integers  $\lambda_1, \dots, \lambda_L$ .

**Proof.** Completely similar to that of the Theorem I.

## 2.1 Example: Sphere

In the  $g = 0$  case the surface that is produced is a sphere with polygonal boundaries. The big decomposition factor in (3) trivialises, and the formula (3) reduces to (all  $n_i > 0$ )

$$\mathcal{N}_{0,L}(n_1, n_2, \dots, n_L) = n_1 \dots n_L \cdot \frac{(\Sigma n + 2L - 3)!}{(\Sigma n + L - 1)!}. \quad (6)$$

For example,

$$\mathcal{N}_{0,1}(n_1) = 1,$$

$$\mathcal{N}_{0,2}(n_1, n_2) = n_1 n_2,$$

$$\mathcal{N}_{0,3}(n_1, n_2, n_3) = n_1 n_2 n_3 \cdot (n_1 + n_2 + n_3 + 3),$$

$$\mathcal{N}_{0,4}(n_1, n_2, n_3, n_4) = n_1 n_2 n_3 n_4 \cdot (n_1 + n_2 + n_3 + n_4 + 4)(n_1 + n_2 + n_3 + n_4 + 5),$$

$$\mathcal{N}_{0,5}(n_1, n_2, n_3, n_4, n_5) = n_1 n_2 n_3 n_4 n_5 \cdot (n_1 + n_2 + n_3 + n_4 + n_5 + 5)(n_1 + n_2 + n_3 + n_4 + n_5 + 6)(n_1 + n_2 + n_3 + n_4 + n_5 + 7).$$

## 2.2 Example: Torus

In the  $g = 1$  case the surface that is produced is a torus with polygonal boundaries. The big decomposition factor in (3) now contributes, and the formula (3) takes the form (all  $n_i > 0$ )

$$\mathcal{N}_{1,L}(n_1, n_2, \dots, n_L) = \frac{1}{4} n_1 \dots n_L \cdot \frac{(\Sigma n + 2L + 1)!}{(\Sigma n + L + 1)!} \cdot \left[ \frac{(n_1 + 1)(n_1 + 2)}{6} + \dots + \frac{(n_L + 1)(n_L + 2)}{6} \right]. \quad (7)$$

For example,

$$\mathcal{N}_{1,1}(n_1) = n_1 (n_1 + 1)(n_1 + 2)(n_1 + 3)/4!,$$

$$\mathcal{N}_{1,2}(n_1, n_2) = n_1 n_2 \cdot (n_1 + n_2 + 4)(n_1 + n_2 + 5)(n_1^2 + n_2^2 + 3n_1 + 3n_2 + 4)/4!,$$

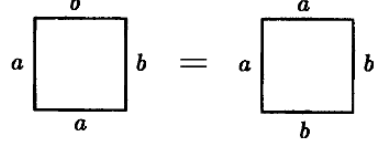
$$\mathcal{N}_{1,3}(n_1, n_2, n_3) = n_1 n_2 n_3 \cdot (n_1 + n_2 + n_3 + 5)(n_1 + n_2 + n_3 + 6)(n_1 + n_2 + n_3 + 7)(n_1^2 + n_2^2 + n_3^2 + 3n_1 + 3n_2 + 3n_3 + 6)/4!,$$

$$\mathcal{N}_{1,4}(n_1, n_2, n_3, n_4) = n_1 n_2 n_3 n_4 \cdot (n_1 + n_2 + n_3 + n_4 + 6)(n_1 + n_2 + n_3 + n_4 + 7)(n_1 + n_2 + n_3 + n_4 + 8)(n_1^2 + n_2^2 + n_3^2 + n_4^2 + 3n_1 + 3n_2 + 3n_3 + 3n_4 + 8)/4!.$$

### 3 The Harer-Zagier case

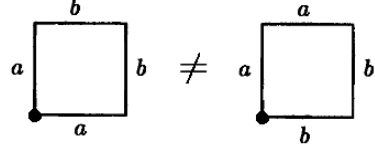
This section is devoted to comparison with the results of [1]. We claim, that  $\mathcal{N}_{g,L}(n_1, n_2, \dots, n_L)$  is a multi-parametric generalization of numbers  $\epsilon_g(N)$ . Let us specify precisely, what particular case corresponds to  $\epsilon_g(N)$ .

Gluing of edges in a polygon naturally possess a cyclic symmetry. For example, the following two complete gluings of a 4-gon into a sphere

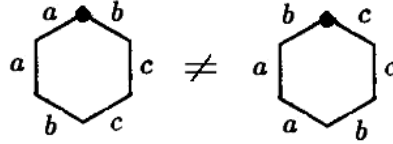


are undistinguishable, because they are related to each other by cyclic exchange of the edges of the square and lead to graphs on the sphere which are related to each other via a diffeomorphism of the sphere.

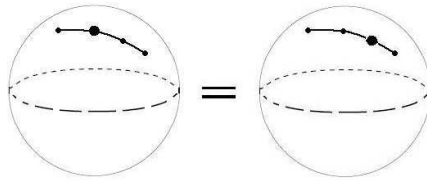
For the problem considered in [1], it was necessary to deal with complete gluings of polygons *without* cyclic symmetry of edges. The simplest way to break the cyclic symmetry is to introduce one distinguished vertex of the polygon:



Then, two gluings in the above example are different. As a result,  $\epsilon_0(2) = 2$ . However, this way of breaking the cyclic symmetry immediately leads to violation of the one-to-one map between the set of equivalence classes of polygon gluings and the set of equivalence classes of graphs on the surfaces. For example, two different gluings



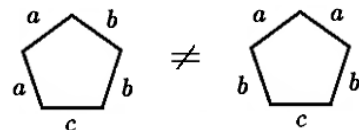
lead to the equivalent graphs on the sphere:



In our problem, we would like to respect the one-to-one map in question. That is why in this paper we do not select any distinguished point of the polygon. Then, complete gluing numbers  $\mathcal{N}_{g,L}(0, 0, \dots, 0)$  do not coincide with the Harer-Zagier numbers.

Instead, it is easy to see that the Harer-Zagier numbers should coincide with  $\mathcal{N}_{g,L}(1, 0, \dots, 0)$ . The presence of single unglued edge breaks the symmetry of glued edges under cyclic exchange. A single unglued edge acts exactly as a distinguished vertex on the polygon's boundary.

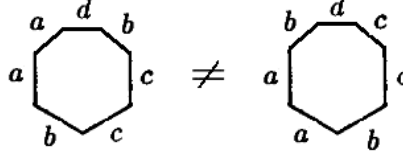
For example, two gluings



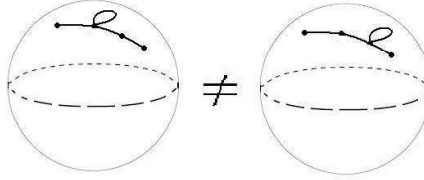
are different, therefore  $\mathcal{N}_{0,3}(1,0,0) = \epsilon_0(2)$ . Generally, the following equality holds:

$$\mathcal{N}_{g,L}(1,0,\dots,0) = \epsilon_g(2g+L-1) \quad (8)$$

In contrast with the distinguished vertex, in the presence of one unglued edge the one-to-one map in question is preserved. For example, two different gluings



lead to two different graphs on the sphere:



Let us make use of the explicit formula for  $\mathcal{N}_{g,L}(n_1, n_2, \dots, n_L)$ . By substituting (5) into (8) and performing the algebra, one obtains an explicit representation of the Harer-Zagier's numbers as the following sum over decompositions:

$$\epsilon_g(N) = \mathcal{N}_{g,N-2g+1}(1,0,\dots,0) = \frac{1}{4^g} \cdot \frac{(2N)!}{(N-2g+1)!(N)!} \cdot \sum_{\lambda_1+\dots+\lambda_L=g} \frac{1}{(2\lambda_1+1)\dots(2\lambda_L+1)}, \quad (9)$$

where  $L = N - 2g + 1$ . For  $g = 0$ , the decomposition factor is trivial (equal to 1) and we recover the Catalan numbers:

$$\epsilon_0(N) = \frac{(2N)!}{(N+1)!(N)!} \cdot \sum_{\lambda_1+\dots+\lambda_L=0} \frac{1}{(2\lambda_1+1)\dots(2\lambda_L+1)} = \frac{(2N)!}{(N+1)!(N)!} = \frac{1}{N+1} \binom{2N}{N}. \quad (10)$$

For  $g = 1$ , the sum can be easily calculated, because all decompositions of 1 into  $L$  parts have a form  $[0, \dots, 1, \dots, 0]$ . In this case, the decomposition factor is equal to  $L/3$  and we obtain

$$\epsilon_1(N) = \frac{(2N)!}{4(N-1)!(N)!} \cdot \sum_{\lambda_1+\dots+\lambda_L=1} \frac{1}{(2\lambda_1+1)\dots(2\lambda_L+1)} = \frac{(2N)! \cdot (N-1)/3}{4(N-1)!(N)!} = \frac{1}{12} \frac{(2N)!}{(N-2)!N!}. \quad (11)$$

Similar calculations can be done for  $g > 1$ . The well-known Harer-Zagier generating function

$$1 + 2 \sum_{g=0}^{\infty} \sum_{N=2g}^{\infty} \epsilon_g(N) x^{N+1} y^{N-2g+1} \frac{1}{(2N-1)!!} = \left( \frac{1+x}{1-x} \right)^y$$

follows from our formula (9) at once:

$$\begin{aligned} & 1 + 2 \sum_{g=0}^{\infty} \sum_{N=2g}^{\infty} \epsilon_g(N) x^{N+1} y^{N-2g+1} \frac{1}{(2N-1)!!} = \\ & = 1 + 2 \sum_{g=0}^{\infty} \sum_{N=2g}^{\infty} x^{N+1} y^{N-2g+1} \frac{1}{4^g} \cdot \frac{(2N)!}{(2N-1)!!(N)!} \frac{1}{(N-2g+1)!} \cdot \sum_{\lambda_1+\dots+\lambda_L=g} \frac{1}{(2\lambda_1+1)\dots(2\lambda_L+1)} = \\ & = 1 + 2 \sum_{g=0}^{\infty} \sum_{N=2g}^{\infty} x^{N+1} y^{N-2g+1} 2^{-2g} 2^N \cdot \frac{1}{(N-2g+1)!} \cdot \sum_{\lambda_1+\dots+\lambda_L=g} \frac{1}{(2\lambda_1+1)\dots(2\lambda_L+1)} = \end{aligned}$$

$$\begin{aligned}
&= 1 + \sum_{g=0}^{\infty} \sum_{L=1}^{\infty} (2x)^L y^L x^{2g} \cdot \frac{1}{L!} \cdot \sum_{\lambda_1+\dots+\lambda_L=g} \frac{1}{(2\lambda_1+1)\dots(2\lambda_L+1)} = \\
&= 1 + \sum_{L=1}^{\infty} \frac{(2y)^L}{L!} \sum_{g=0}^{\infty} \sum_{\lambda_1+\dots+\lambda_L=g} \frac{x^{2g+L}}{(2\lambda_1+1)\dots(2\lambda_L+1)} = \\
&= 1 + \sum_{L=1}^{\infty} \frac{(2y)^L}{L!} \left[ \sum_{\lambda=0}^{\infty} \frac{x^{2\lambda+1}}{(2\lambda+1)} \right]^L = \\
&= 1 + \sum_{L=1}^{\infty} \frac{(2y)^L}{L!} \left[ \frac{1}{2} \log \frac{1+x}{1-x} \right]^L = \exp \left[ 2y \cdot \frac{1}{2} \log \frac{1+x}{1-x} \right] = \left( \frac{1+x}{1-x} \right)^y.
\end{aligned}$$

## 4 Conclusion

Instead of conclusion let us explain what construction is behind the  $\mathcal{N}_{g,L}(n_1, n_2, \dots, n_L)$  numbers. It is well known that cellular decompositions of polygons (with fixed orientation and one distinguished edge) by diagonals are in one-to-one correspondence with Stasheff polytopes (see e.g. [4]). Let us explain the correspondence.

One can draw at the same time  $N - 3$  non-intersecting diagonals in an  $N$ -gon. This way we obtain a triangulation of the polygon. The correspondence in question is pure combinatorial: a  $k$ -dimensional ( $k \leq N - 3$ ) face of a polytop corresponds to the cellular decomposition of the polygon by its diagonals, which can be obtained from the polygon's triangulation via removal of  $k$  among  $N - 3$  diagonals. A  $k$ -dimensional face of the Stasheff polytop belongs to the boundary of an  $n$ -dimensional face ( $N - 3 \geq n > k$ ) if and only if the cellular decomposition with  $N - 3 - n$  diagonals (corresponding to the  $n$ -dimensional face) can be embedded into the cellular decomposition with  $N - 3 - k$  diagonals (corresponding to the  $k$ -dimensional face). E.g. vertices of the Stasheff polytop are diagonal triangulations of the polygon, edges — are triangulations without one diagonal and etc.

If we recall that the polygon topologically is just a disc, one would like to consider a generalization of that construction to real two-dimensional surfaces with handles and polygonal boundaries. I.e. we would like to build combinatorial complexes related to cellular decompositions of real two-dimensional surfaces. So that a  $k$ -dimensional face of such a complex is related to the triangulation of the surface with  $k$  edges of the triangulation removed. Obviously these  $k$ -dimensional faces are just  $k$ -dimensional Stasheff polytopes. From this construction it is clear that any one-face graph on the surface is related to the maximal dimension cell in the corresponding complex. Then,  $\mathcal{N}_{g,L}(n_1, n_2, \dots, n_L)$  counts just the number of cells of the maximal dimension in the complex corresponding to the surface with  $g$  handles and  $L$  polygonal holes with  $n_1, n_2, \dots, n_L$  edges.

In the literature one usually considers ribbon graphs, which are dual to the cellular decompositions, on the real two-dimensional surfaces instead of the cellular decompositions themselves. In that case the complex in question is referred to as graph complex (see e.g. [3]). The important problem is to calculate the homology of this complex.

There are other interesting questions related to the  $\mathcal{N}_{g,L}(n_1, n_2, \dots, n_L)$  numbers: such as finding the generating function for these numbers and/or matrix integral representation. Graphs on surfaces appear in various branches of modern physics, such as matrix models of two-dimensional quantum gravity [5], so an existence of a matrix integral representation for  $\mathcal{N}_{g,L}(n_1, n_2, \dots, n_L)$  would not be surprising.

## 5 Acknowledgements

ETA would like to thank I.Artamkin and S.Lando for valuable discussions and A.Babichev for lessons of mathematical culture and grammar. As well we would like to thank the referee of "Functional Analysis and Applications" journal for very valuable comments and corrections. This work was partially supported by the Federal Nuclear Energy Agency. The work of S.S. was partially supported by the Program for Supporting Leading Scientific Schools (Grant No. NSh-8004.2006. 2), the Russian Foundation for Basic Research (Grant Nos. RFBRItaly 06-01-92059-CE and 07-02-00642) and by the Dynasty Foundation.



## References

- [1] J. Harer, D. Zagier, *The Euler Characteristic of the Moduli Space of Curves I*, Inv. Math. **85**(1986) 457-485;
- [2] S. Lando, *Lectures on generating functions*, MCCME, 2004;  
S. Lando, A. Zvonkin, *Graphs on Surfaces and Their Applications*, Encyclopedia of Mathematical Sciences **141**(2003), Springer;
- [3] M. Kontsevich, *Feynman diagrams and low-dimensional topology*, First European Congress of Mathematics, Vol. II (Paris, 1992) 97121; Progr. Math. **120**(1994), Birkhuser, Basel;
- [4] J. Stasheff, *Homotopy associativity of H-spaces I*, Trans. Amer. Math. Soc. **108**(1963) 275292;  
I. Gelfand, M. Kapranov and A. Zelevinsky, *Discriminants, Resultants and Multidimensional Determinants*, Birkhauser, 1994;
- [5] M. Kontsevich, *Intersection theory on the moduli space of curves*, Funkts. Anal. Prilozh., **25:2**(1991) 5057.